HYPERBOLIC MODIFICATION OF THE NAVIER–STOKES EQUATIONS

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1. A comparison of the speeds of propagation of disturbances in using the Euler and Navier–Stokes models leads to very strange conclusions: whereas in an ideal gas this speed, in complete agreement with experiment, is finite and equal to the speed of sound, in a viscous gas (i.e., in a model called upon to express more accurately the properties of real gases) it is infinite. The latter, naturally, can give rise to certain doubts as to the unconditional validity of the Navier–Stokes equations. The desire then arises to clarify the reasons for this paradox and to attempt to eliminate them.

In this connection we consider first an elementary model, namely, that of a barotropic medium with equation of state $p = a^2\rho$, where a = const is the speed of sound, p is the pressure, and ρ is the density. The equations describing the motion of such a medium have, in the absence of mass forces, the form

$$\begin{gathered}\rho_t + \operatorname{div}(\rho \mathbf{v}) = 0,\\ \rho(\mathbf{v}_t + \mathbf{v}\cdot\operatorname{grad}\mathbf{v}) + \operatorname{grad} p = \operatorname{div}(2\mu D).\end{gathered} \tag{1.1}$$

Here **v** is the speed; D is the deformation velocity tensor; μ is the coefficient of viscosity, equal to zero in the case of an ideal gas (for the same of simplicity we neglect the second viscosity). Equations of the characteristics (in all that follows these will be supplied for the case of one-dimensional motion) for $\mu = 0$ follow from the relation [1]

$$\begin{vmatrix} \omega_t + u\omega_x, & \rho\omega_x \\ a^2\omega_x, & \rho(\omega_t + u\omega_x) \end{vmatrix} = 0,$$

where $\omega(t, x)$ = const is the family of sought-for characteristics; u is the velocity in one-dimensional motion. It follows from this that in the case of an ideal gas there are two families of characteristics, given by the equations

$$\omega_t + u\omega_x = \pm a\omega_x.$$

This means that disturbances (i.e., discontinuities in the density and in the speed of the first and higher orders), if they arise, propagate through the gas with the speed of sound a.

In the case of one-dimensional motion of a viscous gas the characteristics are given by the relation

$$\begin{vmatrix} \omega_t + u\omega_x, & 0 \\ a^2\omega_x, & -2\mu\omega_x^2 \end{vmatrix} = 0,$$

a consequence of which here is that there arises a family of doubled characteristics t = const, i.e., the propagation of certain disturbances takes place instantaneously. But this contradicts both the experimental fact, mentioned above, concerning the propagation of disturbances with the speed of sound and the presently generally accepted statement concerning the impossibility of instantaneous propagation of arbitrary disturbances connected with energy transport.

Thus we conclude that some change in the equations describing the motion of a viscous gas is needed which would lead to a finite speed of propagation of disturbances.

2. It is clear that an error, leading to the instantaneous transmission of disturbances in a viscous gas, is concealed somewhere in the basic assumptions, the basic axioms, used in deriving the equations of motion. We shall analyze, in connection with this, the derivation of equations (1.1) from the conservation laws of mechanics. Moreover, it must be noted that there is no rigorous justification for the validity of the conservation laws of mechanics for continuous (and, in particular,

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viscous) media; these laws are, in essence, merely postulated. The latter means that their applicability is not subject to doubt if and only if the consequences of these laws do not contradict experimental data. In our case a situation even arises in which a consequence of the conservation laws (the infinite speed of propagation of disturbances) contradicts an experimental fact, namely, the propagation of disturbances with the finite speed of sound. Thus, in analyzing the derivation of equations (1.1) we must turn our attention to the conservation laws, particularly, to the situation in which viscosity appears (i.e., we must take into account thermal motion of the molecules).

It is useful to recall the reasoning leading to equations (1.1). We introduce a moving volume τ, the image of which in Lagrangian coordinates, τ_0, is constant. The conservation laws for mass and momentum have the forms

$$\frac{d}{dt}\int_\tau \rho dx = 0, \quad \frac{d}{dt}\int_\tau \rho \mathrm{v} dx = \int_S \sigma\cdot\mathrm{n} dS. \tag{2.1}$$

Here σ is the stress tensor: $\sigma = 2\mu\mathrm{D} - \mathrm{pI}$; S is the surface bounding volume τ; n is the normal to this surface; I is the unit tensor. Next, using the known Euler formulas and the arbitrariness of volume τ, we obtain, subject to corresponding smoothness of the functions, the equations (1.1).

It is not possible, assuming the validity of relations (2.1), to detect any flaws in the reasoning presented; thus, the only way left for "revising" equations (1.1) is to change the formulation of the conservation laws, particularly, the part of them connected with the thermal motion of molecules. Therefore we return to the consideration of a yet simpler problem, namely, heat transfer in a fixed medium. According to the law of conservation of heat, we can write

$$\frac{d}{dt}\int_\tau W dx = \int_S \lambda \frac{\partial W}{\partial n} dS, \tag{2.2}$$

where W is the quantity of heat per unit volume; λ is the coefficient of thermal conductivity; τ is an arbitrary fixed volume. A consequence of equation (2.2) is the well-known thermal conductivity equation

$$W_t = \mathrm{div}\,(\lambda\ \mathrm{grad}\ W). \tag{2.3}$$

Here, as in the case of a viscous gas, propagation of disturbances takes place instantaneously. We turn our attention now to the fact that both in the problem of thermal conductivity and in problems concerning the motion of a viscous gas, an infinite speed of propagation of disturbances arises there where the thermal motion of molecules is taken into account. In connection with this, it is natural to suppose that just in the interpretation of the thermal motion of molecules in the conservation laws of mechanics some error is admitted leading to the paradox involving ana infinite speed of propagation of disturbances.

An analysis of the elementary relationship (2.2) leads to the though that the only possibility for obtaining an equation which handles heat transfer with a finite speed of propagation of disturbances is to consider the right and left sides of relation (2.2) at different instants of time. In other words, it is necessary to write the law for heat conservation in the form

$$\left[\frac{d}{dt}\int_\tau W dx\right]_{t+\alpha} = \left[\int_S \lambda \frac{\partial W}{\partial n} dS\right]_t, \tag{2.4}$$

where the right side is considered at time t and the left side at time t + α ($\alpha > 0$). The latter may be the result of the fact that the process of energy exchange between molecules at the time of collision during thermal motion takes place, not instantaneously, but after a finite interval of time, thus implying a radiation in the variation of W. We remark that systems with a delay of action occur widely in nature. thus, in connection with the gravitational interaction of material bodies the force of interaction is determined by their location, with the time of passage of signal from body to body taken into account, i.e., with retardation, the amount of which is very small for small distances between bodies, but plays an important role in celestial mechanics.

From the conservation law (2.4) we can obtain, for small α, an approximate equation (to within the first power of α inclusive)

$$W_t + \alpha W_{tt} = \mathrm{div}\,(\lambda\ \mathrm{grad}\ W) \tag{2.5}$$

of hyperbolic type, the characteristics for which are well known:

$$\omega_t = \pm \sqrt{\lambda/\alpha}\,\omega_x.$$

Thus, taking into account a retardation in the result of the action in relation (2.4) has enabled us to obtain an equation describing the process of heat propagation without an instantaneous transfer of disturbances.

It should be noted that the question concerning finiteness of the speed of heat propagation has been posted before. Indeed, in [2] attention was called to the need for considering, in certain cases, a finite speed of heat transfer. In [3] the author stated a proposition concerning the boundedness of the rate of diffusion of heat and mass and, developing Prigozhin's principle, he obtained a generalized system of linear Onsager equations, giving rise to hyperbolic equations of heat conduction and diffusion, analogous to those we obtained in revising the conservation law (2.4) into the equation (2.5). In essence, in the derivation of hyperbolic equations of heat and mass transport in [3] a relaxation time was taken into account, i.e., a retardation was introduced as well. For our purposes, consisting in the derivation of refined equations of motion of a viscous gas, it is more convenient to employ the reasoning used in obtaining the relation (2.4).

3. Thus, in agreement with the above, in the derivation of the equations of motion of a viscous gas, one must, in all cases in which thermal motion of molecules is taken into account, introduce a retardation into the formulation of the conservation laws. In particular, the law of conservation of momentum must have the form

$$\left[\frac{d}{dt}\int_{\tau} \rho \mathbf{v}\,dx\right]_{t+\alpha} = \left[\int_{S} \sigma\cdot\mathbf{n}\,dS\right]_{t}$$

or, after the usual transformations,

$$\left[\int_{\tau} \rho\frac{d\mathbf{v}}{dt}\,dx\right]_{t+\alpha} = \left[\int_{\tau} \operatorname{div}\sigma\,dx\right]_{t}, \tag{3.1}$$

where, as usual,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\operatorname{grad}.$$

We note, in contrast to the case of heat propagation in a fixed medium, that the moving volume in equation (3.1) can be different at the times t and t + α; therefore, before going over to a differential equation, as is done in heat transfer, it is necessary to first go to Lagrangian variables ξ^i, recalling that the image of a moving volume in lagrangian variables is one and the same for instants of time. As a result, we obtain

$$\left[\int_{\tau_0} \rho\,\frac{d\mathbf{v}}{dt}\,\frac{\partial(x)}{\partial(\xi)}\,d\xi\right]_{t+\alpha} = \left[\int_{\tau_0} \frac{\partial(x)}{\partial(\xi)}\operatorname{div}\sigma\,d\xi\right]_{t},$$

whence, by virtue of the arbitrariness of τ_0,

$$\left[\rho\,\frac{d\mathbf{v}}{dt}\right]_{t+\alpha}\frac{\partial\,[x(t+\alpha,\xi)]}{\partial(\xi)} = [\operatorname{div}\sigma]_t\,\frac{\partial\,[x(t,\xi)]}{\partial(\xi)},$$

or, after obvious simplifications,

$$\left[\rho\,\frac{d\mathbf{v}}{dt}\right]_{t+\alpha}\frac{\partial\,[x(t+\alpha,\xi)]}{\partial\,[x(t,\xi)]} = [\operatorname{div}\sigma]_t. \tag{3.2}$$

In accordance with the definition of Lagrangian coordinates

$$v^i = \frac{dx^i}{dt},$$

therefore, approximately to within first powers of α inclusive,

$$x^i(t+\alpha,\xi) = x^i(t,\xi) + \alpha v^i(t,\xi),$$

so that, to this degree of accuracy, we have the expression

$$\frac{\partial\,[x\,(t+\alpha,\,\xi)]}{\partial\,[x\,(t,\,\xi)]} = 1 + \alpha\,\mathrm{div}\,\mathbf{v},$$

substituting this expression into equation (3.2) and writing the quantity $\rho[t + \alpha, \xi]$, to within α (as is the case in what follows) in the form

$$\rho\,[t+\alpha,\,\xi\,] = \rho\,(t,\,\xi) + \alpha\,\frac{d\rho}{dt},$$

we obtain from relation (3.2) to within the accuracy indicated

$$\rho\,(t,\,x)\left[\frac{d\mathbf{v}}{dt}\right]_{t+\alpha} = [\mathrm{div}\,\sigma\,]_t, \tag{3.3}$$

or, equivalently,

$$\rho\left[\frac{d\mathbf{v}}{dt} + \alpha\,\frac{d^2\mathbf{v}}{dt^2}\right] = \mathrm{div}\,\sigma.$$

Here the right and left sides of the equation are considered for one and the same instant of time. Thus the complete refined system of equations describing the motion of the gas considered has, in the absence of mass forces, the form

$$\begin{gathered}\rho_t + \mathrm{div}\,(\rho\mathbf{v}) = 0, \quad \rho\left[\frac{d\mathbf{v}}{dt} + \alpha\,\frac{d^2\mathbf{v}}{dt^2}\right] = \mathrm{div}\,\sigma,\\ \sigma = 2\mu D - pI, \quad p = a^2\rho, \quad \mu = \mu\,(\alpha,\,\rho).\end{gathered} \tag{3.4}$$

Equations of the characteristics in the one-dimensional case follow from the relation

$$\begin{vmatrix} \omega_t + u\omega_x, & 0 \\ a^2\omega_x - 2\,\dfrac{d\mu}{d\rho}\,u_x\omega_x, & \alpha\rho\,(\omega_t + u\omega_x)^2 - 2\mu\omega_x^2 \end{vmatrix} = 0,$$

so that here we have three families of characteristics:

$$\omega_t + u\omega_x = 0, \quad \omega_t + u\omega_x = \pm\sqrt{2\mu/\alpha\rho}\,\omega_x.$$

Putting

$$2\mu = \alpha\rho a^2, \tag{3.5}$$

we find that even for a viscous gas propagation of disturbances with respect to the moving gas will take place with a finite sped, namely, the speed of sound. It is obvious that as α tends towards zero the equations (3.4), subject to the condition (3.5), will go over into equations (1.1) for $\mu = 0$.

We now consider questions relating to the formulation of the simplest initial-boundary problems for one-dimensional equations. It is natural to require that problems for equations (3.4) pass over into corresponding problems for equations (1.1) as α tends towards zero. It is easy to see that in this and the other case there are two flow regimes, differing in their formulations of the initial-boundary problems: the subsonic and the supersonic. We limit our discussion only to the statement of the problem for supersonic flow. As to the region in which a solution is sought, we consider the set of planar points t, x, specified by the inequalities $0 < t < T$, $0 < x < l$. In this case the initial-boundary problem for the one-dimensional flow of a gas for $u > a$ for equations (1.1) can be formulated thus:

$$\begin{aligned} &u = \varphi\,(x) > 0, \quad \rho = \psi\,(x) > 0, \quad t = 0, \quad 0 \leqslant x \leqslant l,\\ &u = U\,(t) > 0, \quad \rho = V\,(t) > 0, \quad 0 < t \leqslant T, \quad x = 0.\end{aligned} \tag{3.6}$$

For the equations (3.4) the number of conditions on the boundary must naturally be increased; however, the additional conditions must be formulated in such a way that as α tends towards zero they will be a consequence of conditions (3.6) and the equations (1.1). The latter can be guaranteed by specifying the additional conditions in the form

$$u_t = f(\alpha, x) - \varphi\varphi_x - a^2\psi_x/\psi, \quad t = 0, \quad 0 \leqslant x \leqslant l;$$
$$U_t(t)\, u_x = F(\alpha, t) - \frac{dU}{dt} - a^2\rho_x/V(t), \quad 0 < t < T, \quad x = 0.$$

Here the functions f(α, x), F(α, t) must satisfy the requirements

$$\lim_{\alpha\to 0} f(\alpha, x) = 0, \quad 0 \leqslant x \leqslant l, \quad \lim_{\alpha\to 0} F(\alpha, t) = 0, \quad 0 < t < T.$$

It is not difficult to see that in this case the initial-boundary problem for equations (3.4) passes over, as α tends towards zero, into the problem for equations (1.1). One can expect also that the solution of the problem concerning the flow of a viscous gas will pass over into the solution of the problem concerning the flow of an ideal gas.

The following question arises: when must the revised equations (3.4) be used instead of the usual equations (1.1)? To answer this, we carry out, as is customary, a change-over to dimensionless variables, introducing a characteristic length l, a characteristic speed, namely, the speed of sound a, and a characteristic time θ, depending on the frequency, from which it is necessary to obtain information concern the process. As a result, in dimensionless variables equations (3.4) are representable in the form

$$\rho_t + B\,\mathrm{div}\,(\rho \mathbf{v}) = 0,$$
$$\rho\left[\frac{dv}{dt} + A\frac{d^2\mathbf{v}}{dt^2}\right] + B\,\mathrm{grad}\,\rho = AB^2\,\mathrm{div}\,(\rho D), \tag{3.7}$$

where the dimensionless parameters A, B have the form

$$A = \alpha/\theta, \quad B = a\theta/l,$$

for which

$$\frac{d}{dt} = \frac{\partial}{\partial t} + B\mathbf{v}\cdot\mathrm{grad}.$$

It is evident from an analysis of equations (3.7) that the usual equations can be employed when A << 1; however B >> 1, so that the quantity AB^2 becomes essential. In other words, equations (1.1) can be used in those cases when the frequency with which it is necessary to extract information is small ($\theta >> \alpha$) and the speed of sound is large in comparison with the quantity l/θ. However, even when these conditions are satisfied the revised equations enable us to extract certain interesting properties of viscous gas flows. We consider, in this connection, a problem concerning the dissipation of mechanical energy. For the sake of convenience, we shall label quantities calculated at time t + α an asterisk-subscript, reserving the usual notation for quantities calculated at time t. From the law (3.3) for the conservation of momentum we find, after multiplication by v$_*$,

$$\rho\left[\frac{d}{dt}\frac{\mathbf{v}^2}{2}\right]_* = \mathbf{v}_*\cdot\mathrm{div}\,\sigma = (\mathbf{v}\cdot\mathrm{div}\,\sigma) + \alpha\frac{d\mathbf{v}}{dt}\cdot\mathrm{div}\,\sigma,$$

or, what amounts to the same thing, to within quantities of order α inclusive,

$$\rho\left[\frac{d}{dt}\frac{\mathbf{v}^2}{2}\right]_* = \mathrm{div}\,(\sigma\cdot\mathbf{v}) - \sigma : D + \alpha\rho\frac{d\mathbf{v}}{dt}\cdot\frac{d\mathbf{v}}{dt}. \tag{3.8}$$

After obvious simplifications this can be rewritten as

$$\rho\left[\frac{d}{dt}\frac{\mathbf{v}^2}{2}\right]_* = \alpha\rho\frac{d\mathbf{v}}{dt}\cdot\frac{d\mathbf{v}}{dt} - 2\mu D : D + \mathrm{div}\,(2\mu D\cdot\mathbf{v}) - \mathbf{v}\cdot\mathrm{grad}\,p.$$

Thus, to the dissipative Rayleigh function there is here added a new quantity of opposite sign. In this connection it is completely possible that at some points of the flow region where large accelerations arise the quantity

$$\Phi = \rho\alpha\frac{d\mathbf{v}}{dt}\cdot\frac{d\mathbf{v}}{dt} - 2\mu D : D$$

can be positive. From the point of view of the usual equations the latter can be interpreted as a manifestation of negative viscosity [4].

It is possible that the appearance of flow regions for which $\Phi > 0$, has an essential influence on flow stability and may be the reason for the rise of turbulence.

4. It is convenient here to supply the revised equations of motion of a viscous gas in the most general case for an arbitrary relationship between stresses and deformations and for an arbitrary equation of state. Equations (3.4) for the conservation of mass and momentum are valid even in the most general case if in the equation for the conservation of momentum we add-in the mass forces. In this way it turns out to be sufficient to obtain the just revised equation of energy. Denoting the internal energy per unit mass of gas by ε, we can write the law of conservation of energy in the moving volume, taking retardation into account, in the absence of external forces, in the following form:

$$\left[\frac{d}{dt}\int_{\tau} \rho \left(\varepsilon + \frac{v^2}{2}\right) dx\right]_* = \int_S \mathbf{v}\cdot\sigma\cdot\mathbf{n}dS. \tag{4.1}$$

Here the starred quantity is calculated at the time $t + \alpha$. After carrying out arguments analogous to those described in deriving the equations for conservation of momentum, we obtain

$$\rho\left[\frac{d}{dt}\left(\varepsilon + \frac{v^2}{2}\right)\right]_* = \operatorname{div}(\sigma\cdot\mathbf{v}).$$

Substituting here from equation (3.8) the expression for $\rho\left[\frac{d}{dt}\frac{v^2}{2}\right]_*$ and bringing in similar expressions, we write the equation for the conservation of energy in the form

$$\rho\left[\frac{d\varepsilon}{dt}\right]_* = \sigma : D - \rho\alpha\frac{d\mathbf{v}}{dt}\cdot\frac{d\mathbf{v}}{dt}.$$

Thus, in the most general case, we write the revised equations of motion of a gas as follows:

$$\begin{gathered}\rho_t + \operatorname{div}(\rho\mathbf{v}) = 0, \quad \rho\left(\frac{d\mathbf{v}}{dt} + \alpha\frac{d^2\mathbf{v}}{dt^2}\right) = \operatorname{div}\sigma, \\ \rho\left(\frac{d\varepsilon}{dt} + \alpha\frac{d^2\varepsilon}{dt^2}\right) = \sigma : D - \rho\alpha\frac{d\mathbf{v}}{dt}\cdot\frac{d\mathbf{v}}{dt}.\end{gathered} \tag{4.2}$$

For completeness we need to add to these equations a canonical equation of state, i.e., a relationship between thermodynamic characteristics of the medium, and a defining equation specifying the relationship between stresses and deformations, taking thermal motion of the molecules and viscosity into account.

We show that for a reasonable choice of closure equations the speed of propagation of disturbances in the case of equations (4.2) is finite. It is convenient to write the equation of state of a simple medium in the form

$$p = p(\varepsilon, \rho),$$

and we specify the relationship between stresses and deformations by the expression

$$\sigma = 2\mu D + (\zeta \operatorname{div}\mathbf{v} - p) I,$$

where ζ is the second viscosity. We write the equations for the one-dimensional motion of such a gas, in the absence of mass forces, in the form

$$\rho_t + u\rho_x + \rho u_x = 0,$$

$$\rho\left[\frac{du}{dt} + \alpha\frac{d^2u}{dt^2}\right] = \frac{\partial}{\partial x}\left[(2\mu + \zeta)\frac{\partial u}{\partial x}\right] - \frac{\partial p}{\partial\varepsilon}\frac{\partial\varepsilon}{\partial x} - \frac{\partial p}{\partial\rho}\frac{\partial\rho}{\partial x},$$

$$\rho\left[\frac{d\varepsilon}{dt}+\alpha\frac{d^2\varepsilon}{dt^2}\right]=(2\mu+\zeta)\left(\frac{\partial u}{\partial x}\right)^2-p\frac{\partial u}{\partial x}-\rho\alpha\left(\frac{du}{dt}\right)^2, \tag{4.3}$$

$$\frac{d}{dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x},\quad (2\mu+\zeta)=\alpha f(\varepsilon,\rho).$$

The characteristics of this system of equations are given by the relation

$$\begin{vmatrix} \omega_t+u\omega_x, & 0, & 0 \\ \left(\frac{\partial p}{\partial \rho}+\alpha\frac{\partial f}{\partial \rho}\frac{\partial u}{\partial x}\right)\omega_x, & \alpha\rho(\omega_t+u\omega_x)^2-(2\mu+\zeta)\,\omega_x^2, & 0 \\ 0, & 0, & \alpha\rho(\omega_t+u\omega_x)^2 \end{vmatrix}=0.$$

It follows from this that we have here a three-fold characteristic

$$\omega_t+u\omega_x=0 \tag{4.4}$$

and two characteristics defined by the equations

$$\omega_t+u\omega_x=\pm\sqrt{(2\mu+\zeta)/\rho\alpha}\,\omega_x. \tag{4.5}$$

Putting

$$2\mu+\zeta=\alpha\rho a^2(\varepsilon,\rho),$$

we find, by virtue of equations (4.5), that disturbances here are propagated with a finite speed, namely, the speed of sound a.

In the case of an ideal gas, i.e., for $\alpha=0$, the equations for one-dimensional motion (for an arbitrary equation of state) have characteristics given by the equations

$$\omega_t+u\omega_x=\pm\sqrt{\frac{\partial p}{\partial \rho}+\frac{p}{\rho}\frac{\partial p}{\partial \varepsilon}}\;\omega_x,$$

and a one-fold characteristic of the form (4.4), so that in a viscous gas, in contrast to an ideal gas, there is appended, in addition, a two-fold characteristic (4.4).

An initial-boundary problem for the one-dimensional motion of an ideal gas with an arbitrary equation of state is posed, in the absence of strong discontinuities, in exactly the same way as for a barotropic gas with the addition of obvious conditions making it possible to determine the internal energy of the gas. However, in the case of a viscous gas (i.e., for $\alpha\neq 0$) yet additional conditions must be appended of the form

$$\begin{aligned} &\rho\frac{d\varepsilon}{dt}+p(\varepsilon,\rho)\frac{\partial u}{\partial x}=\alpha A(x), && t=0,\quad 0\leqslant x\leqslant l; \\ &\rho\frac{d\varepsilon}{dt}+p(\varepsilon,\rho)\frac{\partial u}{\partial x}=\alpha B(t), && x=0,\quad 0<t<T; \\ &\rho(u_t+uu_x)+a^2\rho_x=\alpha f(x), && t=0,\quad 0\leqslant x\leqslant l; \\ &\rho(u_t+uu_x)+a^2\rho_x=\alpha F(t), && x=0,\quad 0<t<T, \end{aligned} \tag{4.6}$$

where ρ, u, ε, and their derivatives along corresponding boundaries are calculated from the conditions (3.6).

It is of interest to consider the situation in which strong discontinuities arise in a gas (in our case this involves discontinuities of functions and of some first derivatives). We rewrite the equations of motion of a gas with arbitrary equations of state in the divergent form

$$\rho_t+\operatorname{div}(\rho \mathrm{v})=0,$$

$$\frac{\partial}{\partial t}\left(\rho\left(\mathrm{v}+\alpha\frac{d\mathrm{v}}{dt}\right)\right)+\operatorname{div}\left(\rho\mathrm{v}\left(\mathrm{v}+\alpha\frac{d\mathrm{v}}{dt}\right)-\sigma\right)=0, \tag{4.7}$$

$$\frac{\partial}{\partial t}\left(\rho\left(\varepsilon+\frac{v^2}{2}\right)+\alpha\rho\frac{d}{dt}\left(\varepsilon+\frac{v^2}{2}\right)\right)+\operatorname{div}\left(\rho\mathbf{v}\left(\varepsilon+\frac{v^2}{2}+\alpha\frac{d}{dt}\left(\varepsilon+\frac{v^2}{2}\right)\right)-\mathbf{v}\cdot\sigma\right)=0.$$

It follows from this that the conservation laws at a strong discontinuity (compatibility conditions) can be written in the form

$$\begin{gathered} n^0\,[\rho]+n^i\,[\rho v_i]=0,\\ n^0\left[\rho\left(v_j+\alpha\frac{dv_j}{dt}\right)\right]+n^i\left[\rho v_i\left(v_j+\alpha\frac{dv_j}{dt}\right)-\sigma_{ij}\right]=0,\\ n^0\left[\rho\left(\varepsilon+\frac{v^2}{2}\right)+\alpha\rho\frac{d}{dt}\left(\varepsilon+\frac{v^2}{2}\right)\right]+n^i\left[\rho v_i\left(\varepsilon+\frac{v^2}{2}+\alpha\frac{d}{dt}\left(\varepsilon+\frac{v^2}{2}\right)\right)-v^j\sigma_{ij}\right]=0, \end{gathered}\tag{4.8}$$

where the symbol [...], as usual, indicates a discontinuity of corresponding quantities at a jump; n^0 is the cosine of the angle made by the normal to the discontinuity surface with the time axis; n^i are the cosines of the angles made by the normal with the spatial axes; v_i are velocity components. It is obvious that when $\alpha = 0$ equations (4.7) become the corresponding divergent equations for an ideal gas and the relations (4.8) become the usual compatibility conditions.

In the case of a one-dimensional motion of a viscous gas with defining equation

$$\sigma = 2\mu D + (\zeta\,\operatorname{div}\mathbf{v} - p)\,I;\quad (2\mu+\zeta)=\rho\alpha a^2(\varepsilon,\rho)$$

the compatibility conditions (4.8) assume the form

$$\begin{gathered} [\rho D_*]=0,\quad [p-\rho u D_*]=\left[\alpha\rho\left(a^2u_x+D_*\frac{du}{dt}\right)\right],\\ \left[pu-\rho D_*\left(\varepsilon+\frac{u^2}{2}\right)\right]=\left[\alpha\rho\left(a^2uu_x+D_*\frac{d}{dt}\left(\varepsilon+\frac{u^2}{2}\right)\right)\right]. \end{gathered}\tag{4.9}$$

Here D_* is the rate of travel of the shock wave relative to the gas:

$$D_* = dx/dt - u.$$

Examples can be given for the exact solution of the equations of motion of a viscous gas in the presence of a strong discontinuity. As one such example, we point out the well-known solution relating to the motion of a piston in a gas with constant supersonic speed. In this case, from the instant that motion commences in a quiescent gas, a strong discontinuity propagates, whereby the gas preceding this discontinuity is quiescent while that behind it moves with a constant speed, namely, the speed of the piston. This means that on both sides of the discontinuity derivatives of the functions ε, u are identically equal to zero, i.e., the compatibility conditions (4.9) are satisfied if we take the functions ρ, u, ε from the well-known solution of this problem for an ideal gas.

We shall not consider here questions relating to the formulation and solvability of problems for equations (4.3) in the many-dimensional case.

In an analogous manner we can obtain hyperbolic equations of motion of a viscous gas with heat conduction and diffusion taken into account.

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